The group of diffeomorphisms of $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, denoted by $\mathrm{Diff}(S^1)$, is infinite dimensional Lie group. Corresponding Lie algebra:

$$A = \left\{ f(z) \frac{d}{dz} \;\middle|\; f(z) \in \mathbb{C}[z, z^{-1}] \right\}$$

where $\mathbb{C}[z, z^{-1}]$ is $\mathbb{C}$ algebra of Laurent polynomials. Lie bracket:

$$\left[f(z) \frac{d}{dz}, \; g(z) \frac{d}{dz} \right] = \left\{ f(z) g'(z) - g(z) f'(z) \right\} \frac{d}{dz}$$

Basis generators:

$$L_n = -z^{n+1} \frac{d}{dz}, \qquad n \in \mathbb{Z}$$

with restriction to S^1 given by:

$$L_n = i e^{in\theta} \frac{\partial}{\partial \theta}, \qquad z = e^{i\theta}$$

Commutation relation: $[L_m, L_n] = (m-n) L_{m+n}$

L_n generates infinitesimal transformations of $\mathbb{C} \setminus \{0\}$ given by $\varphi_t(z) = z - t z^{n+1}$

L_0, L_1, L_{-1} extend to $\mathbb{CP}^1$:

$$[L_0, L_1] = -L_1, \quad [L_0, L_{-1}] = L_{-1}, \quad [L_1, L_{-1}] = 2L_0$$

$\longrightarrow$ generate $sl_2(\mathbb{C})$

Proposition 2:

$H^2(A,\mathbb{C}) \cong \mathbb{C}$. The cohomology $H^2(A,\mathbb{C})$ has a basis represented by the 2-cocycle

$$\alpha := \omega\left(f\frac{d}{dz}, g\frac{d}{dz}\right) = \frac{1}{12}\operatorname{Res}_{t=0}(f''' g\, dz)$$

Proof:

Let α be a 2-cocycle of A. Put $\alpha_{p,q} = \alpha(L_p, L_q)$

2-cocycle condition for (L_0, L_p, L_q)

$\xrightarrow[\text{proof of Prop. 1}]{\text{similar to}}$ α is invariant under rotation

also $\alpha_{p,q} = 0$ if $p+q \neq 0$

set $\alpha_p := \alpha_{p,-p} \longrightarrow$ 2-cycle condition:

$$(p+2q)\alpha_q - (2p+q)\alpha_q = (p-q)\alpha_{p+q}$$

(exercise)

$\Rightarrow \alpha_p = \lambda p^3 + \mu p$

We can get μp from

$$d\beta(L_p) = \beta([L_p, L_{-p}]) = 2p\beta(L_0)$$

by setting $\mu := \frac{1}{2}\beta(L_0)$

$\longrightarrow \mu p$ is coboundary and does not change $H^2(A,\mathbb{C})$. Set $\lambda = \frac{1}{12}$, $\mu = -\frac{1}{12}$ $\square$

Recall:

Let M be a smooth manifold and let L be a complex line bundle over M. Fix connection ∇ on L. Then

$$\nabla = d - 2\pi\sqrt{-1}\,\alpha_U \quad \text{locally on } U \subset M$$

$\rightarrow$ $d\alpha_{U_\lambda}$ defines a global 2-form for open covering $\{U_\lambda\}, \lambda \in \Lambda$ of M

$\rightarrow$ first Chern form of ∇: $c_1(\nabla)$

$c_1(\nabla)$ defines a class in $H^2(M, \mathbb{R})$ in the image of $H^2(M, \mathbb{Z})$ $\rightarrow$ first Chern class of L.

Now:

Suppose M is simply connected and fix a base point $x_0 \in M$.

Let $\gamma : [0,1] \rightarrow M$ be a smooth loop with

$$\gamma(0) = \gamma(1) = x_0 .$$

Then $\gamma^*(L)$ $\rightarrow$ complex line bundle on $[0,1]$ with connection $\gamma^*\nabla$.

"horizontal section" s: $(\gamma^*\nabla)s = 0$

For u in fibre of γ^*L at 0 select section s s.t. $s(0) = u$.

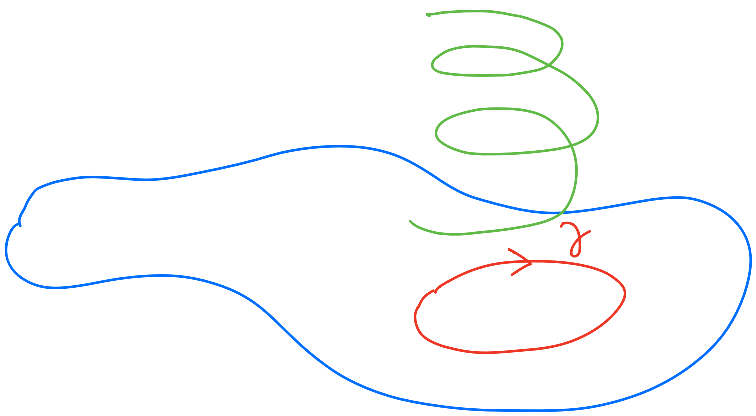
We have dic $D \subset M$ with $\partial D = \gamma$ (M simply connected)

$\longrightarrow$ Stokes theorem gives

$$(*) \qquad S(1) = u \exp\left(2\pi\sqrt{-1}\int_{D_t} c_1(\nabla)\right).$$

Denote by L_{x_0} the fibre of L over x_0.

$\longrightarrow$ $(*)$ gives linear transformation of L_{x_0} denoted "holonomy" of ∇ around γ.



<u>Proposition 3:</u>

Let M be a simply connected smooth manifold and ω a closed 2-form on M with $\omega \in \operatorname{Im} i$ where

$$i : H^2(M,\mathbb{Z}) \rightarrow H^2(M,\mathbb{R})$$

Then $\exists$ complex line bundle L over M and connection ∇ on L s.t.

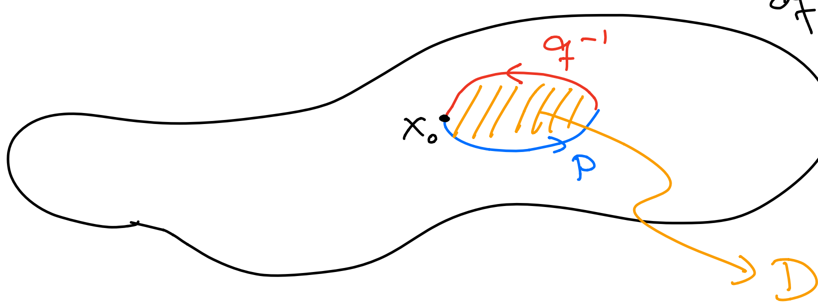
$$c_1(\nabla) = \omega$$

Proof:

Denote by $P_{x_0}(M)$ the set of smooth paths
$$p : [0,1] \longrightarrow M \quad \text{with} \quad p(0) = x_0$$

equivalence relation $\sim$ on $P_{x_0}(M) \times \mathbb{C}$:
$$(p,u) \sim (q,v) \quad \text{iff}$$
$$p(1) = q(1) \quad \text{and} \quad v = u \exp\left(2\pi\sqrt{-1}\int_D \omega\right)$$

independent of D as $\omega \in H^2(M,\mathbb{Z})$



Define $L \equiv P_{x_0}(M) \times \mathbb{C}/\sim$ and proj. map

$\pi : L \longrightarrow M$ by $\pi(p,u) = p(1)$

$\longrightarrow$ connection ∇ of L has holonomy
$$\exp\left(2\pi\sqrt{-1}\int_D \omega\right) \quad \text{around loops } \gamma$$

$\longrightarrow \quad c_1(\nabla) = \omega$

$\square$

Example:

consider $M = LG_{\mathbb{C}}$ and $\omega \in H^2(LG_{\mathbb{C}}, \mathbb{Z})$

Take $G = SU(2)$ and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$

Define for $X \in SU(2)$ 1-form $\mu = X^{-1}dX$

("Maurer-Cartan form") of Lie group $SU(2)$.

Then $\sigma = \frac{1}{24\pi^2} \mathrm{Tr}(\mu \wedge \mu \wedge \mu)$ is generator of $H^3(SU(2), \mathbb{Z})$. Let $\phi : LG \times S^1 \to G$ be given by $\phi(\gamma, z) = \gamma(z)$. Define

$$\omega = - \int_{S^1} \phi^* \sigma$$

It can be shown that $H^2(LG, \mathbb{Z}) \cong \mathbb{Z}$ with generator ω. Associated complex line bundle L is called "fundamental line bundle" over $LG_{\mathbb{C}}$.

§2 Representations of affine Lie algebras

Let g be a complex Lie algebra and V a complex vector space.

Definition:

We call a linear map $\rho : g \to \mathrm{End}(V)$ s.t.

$$\rho([X,Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X)$$

$\forall \; X, Y \in g$ a "linear representation" of g on V.

We will also simply write Xv for $\rho(X)v$.

Focus on $g = sl_2(\mathbb{C}) \longrightarrow \hat{g} = \hat{A}_1$

Lie-algebra: $[H,E] = 2E$, $[H,F] = -2F$, $[E,F] = H$

$\langle H, H \rangle = 2$, $\langle E, F \rangle = 1$, $\langle H, E \rangle = 0 = \langle H, F \rangle$

Denote by $\{I_n\}$ orthonormal basis with respect to $\langle , \rangle$. Then the "Casimir element" C in $\mathcal{U}(g)$ is given by $C = \sum_{\mu} I_{\mu} I_{\mu}$

universal enveloping alg

or $C = \frac{1}{2} H^2 + EF + FE$ (angular momentum operator)

<u>Definition</u> (highest weight representation):

V is "highest weight rep" if:

(a) $\exists\, v \in V$ non-zero s.t. $Hv = \lambda v$ and $Ev = 0$

(b) V is generated by $F^n v$, $n = 0, 1, \dots$

If $F^n v$, $n = 0, 1, \dots$, are linearly independent, V is called "Verma module" M_λ.

For $\lambda \in \mathbb{C}$ generic, M_λ is irreducible g-module. But for $\lambda = 2j$, $j \in \mathbb{Z}$ M_λ becomes reducible and we have sub-rep V_λ spanned by:

$$u_m, \quad m = j, j-1, \dots, -j+1, -j$$

$$H u_m = 2m\, u_m,$$

$$E u_m = \sqrt{(j+m+1)(j-m)}\, u_{m+1}$$

$$F u_m = \sqrt{(j+m)(j-m+1)}\, u_{m-1}$$

In particular: $F^{\lambda+1} u_j = 0$. We have

$$V_\lambda = M_\lambda / F^{\lambda+1} v \quad \text{"spin } j \text{ rep."}$$

C acts as scalar on V_λ s.t.

$$C u_j = 2j(j+1)$$

<u>Next:</u> Representations of affine Lie algebras

Denote by

- $A_+ \subset \mathbb{C}((t))$: the subalgebra $\sum_{n>0} a_n t^n$
- A_- : sb. alg. $\sum_{n<0} a_n t^n$

Define

- $N_+ = [g \otimes A_+] \oplus \mathbb{C}E$,
- $N_0 = \mathbb{C}H \oplus \mathbb{C}c$,
- $N_- = [g \otimes A_-] \oplus \mathbb{C}F$

$\rightarrow \quad \hat{g} = N_+ \oplus N_0 \oplus N_-$

<u>Definition:</u>

Let k and λ be complex numbers. A left $\hat{g}$-module $\hat{V}_{k,\lambda}$ is "highest weight rep." with level k and highest weight λ if:

(a) $\exists\ v \in \hat{V}_{k,\lambda}$ non-zero with

$$N_+ v = 0, \quad cv = kv, \quad Hv = \lambda v$$

(b) $U(N_-)$ generated by v coincides with $\hat{V}_{k,\lambda}$

(a) $\Rightarrow \ cu = ku \ \forall\ u \in \hat{V}_{k,\lambda}$